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Solitary wave, soliton and shelf evolution over variable depth

By R. S. JOHNSON

Department of Mathematics and Statistics, The University of Newcastle upon Tyne, Newcastle upon Tyne, NE1 7RU, UK

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The familiar problem of the propagation of surface waves over variable depth is reconsidered. The surface wave is taken to be a slowly evolving nonlinear wave (governed by the Korteweg–de Vries equation) and the depth is also assumed to be slowly varying; the fluid is stationary in its undisturbed state. Two cases are addressed: the first is where the scale of the depth variation is longer than that on which the wave evolves, and the second is where it is shorter (but still long). The first case corresponds to that discussed by a number of previous authors, and is the problem which has been approached through the perturbation of the inverse scattering transform method, a route not followed here. Our more direct methods reveal a new element in the solution: a perturbation of the primary wave, initiated by the depth change, which arises at the same order as the left-going shelf. The resulting leading-order mass balance is described, with more detail than hitherto (made possible by the use of a special depth variation). The second case is briefly presented using the same approach, and some important similarities are noted.

1. Introduction

The study of both solitary waves and variable depth (and width) has a long history; the seminal work of Boussinesq (1871) and Rayleigh (1876) on the theory of the solitary wave, and the early observations of John Scott Russell (1844), are recognized landmarks. Perhaps only slightly less familiar is the work of Green (1837; see also Lamb 1932, Art. 185) on the effects of variable depth and width; he left as a legacy his 'Green's Law' (which we shall mention in due course). Boussinesq (1872) also examined the slow evolution of the amplitude of a solitary wave as it enters a region of gradual depth change. However, we now see that the introduction of the Korteweg–de Vries (KdV) equation (Korteweg & de Vries 1895) was a major step forward in our understanding of nonlinear wave propagation (Gardner *et al.* 1967).

Although only fairly simple systems give rise to the KdV (or similar) equations, many problems are in some sense approximately KdV-like (but therefore non-integrable). We shall describe one such here: the problem of variable depth. The extension of the KdV equation to a variable-depth regime was given by Johnson (1973*a*) and Kakutani (1971), and by viewing the depth change as either slow or fast, some attempt has been made to obtain asymptotic solutions of the appropriate KdV equation; see Grimshaw (1970, 1971), Johnson (1972, 1973*a, b*) and Leibovich & Randall (1973). However, the important advances were made (in the case of slow depth change) by constructing the inverse scattering transform theory for a perturbed (integrable) system. The perturbed KdV equation may then be treated as a special example of this method; see Kaup & Newell (1978), Newell (1978), Karpman &

Maslow (1977), Candler & Johnson (1981). These papers describe the important discovery that, as a solitary wave propagates, a shelf (of small amplitude) is created behind the wave; this extends back to where linear right-going disturbances would have reached after being initiated by the depth change. Some details as to the form and evolution of this shelf are given by Newell (1978), Miles (1979) and Knickerbocker & Newell (1980).

One result of these calculations is to show that the solitary wave plus the shelf carry $O(1)$ mass, but that this is not the mass (to leading order) which is carried by the incident wave propagating from the left (initially over constant depth, say). Knickerbocker & Newell (1985) introduce the additional component (in the complete water-wave problem) of a reflected wave, i.e. one which propagates to the left. The appearance of a reflected wave is not a new observation in this type of analysis. In a closely related problem, Peregrine (1967) described the left-going (reflected) wave after he had isolated it from his numerical results. He also provided a description of it, based on a suitable set of linearized equations. This wave, although of small amplitude – much smaller than the right-going shelf – can be constructed to carry the appropriate mass which gives the overall mass balance (to leading order). This left-going wave component extends from behind the solitary wave to where linear left-going disturbances would have reached after being initiated by the depth change; its character is also shelf-like. There can be little doubt that this general structure gives an essentially correct description (in terms of leading-order mass-carrying components) of this problem, although we shall present both an important new refinement and some new details.

We shall describe the solution in the form of a systematic asymptotic expansion. Of course, this approach will reproduce some of the results described by Knickerbocker & Newell (1985), but in a manner which, we believe, makes the various elements of the solution quite plain. In addition, we take the opportunity to present some analytical details that are new by taking advantage of a special depth variation, of the form $(1+\alpha Y)^{4/3}$. Above all, we shall see that it is possible to have – indeed it is almost always present – a right-going component (which carries $O(1)$ mass) appearing at the same order as the important left-going shelf, a possibility apparently overlooked by previous authors.

Most of this type of work has been directed towards problems where the scale on which the depth varies is much longer than the scale on which the nonlinear wave evolves. (Of course, this is an essential requirement if the main thrust of the analysis is to be provided by the perturbation of the inverse scattering theory (IST).) Here, we shall modify our approach to the situation where the depth changes slowly, but faster than the evolution scale of the nonlinear wave. Results corresponding to those for very slow depth change are presented and we find, for example, that the general structure of the solution (incident wave, shelves and new perturbation, reflection) is rather similar, although there are some important differences in detail.

2. Governing equations

We model the fluid as inviscid and incompressible, and the flow is irrotational. The ambient state of the fluid is stationary under the action of constant gravitational acceleration. At the free surface we assume that the pressure is constant and we ignore the effects of surface tension. The flow is described in the (x', z')-plane where the bottom boundary of the fluid ($z' = b'(x')$) is a rigid impermeable surface. This configuration, together with the physical (dimensional) variables, is shown in figure 1.

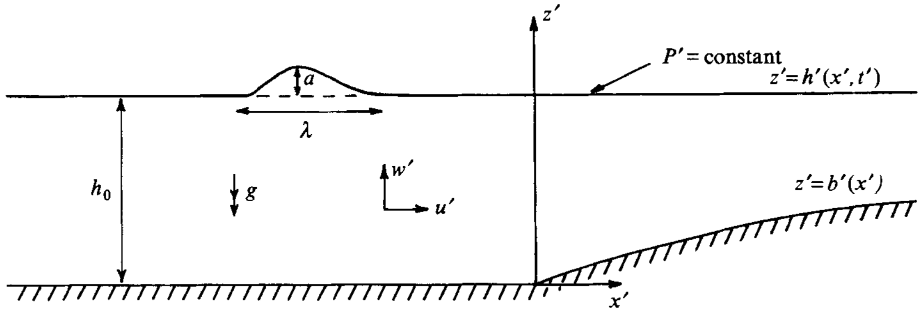

FIGURE 1. The configuration and dimensional variables.

Written in the usual notation, the governing equations – we prefer to work with the Euler equation – are

$$\frac{\mathrm{D}\boldsymbol{u}'}{\mathrm{D}t'} = -\frac{1}{\rho'}\nabla P' + \boldsymbol{g}'; \quad \nabla\cdot\boldsymbol{u} = 0,$$

with $w' = Dh'/\mathrm{D}t'$, P' = constant on $z' = h'(x', t')$; $w'/u' = \mathrm{d}b'/\mathrm{d}x'$ on $z' = b'(x')$. The dimensional variables used here are represented by a prime, with $\boldsymbol{u}' \equiv (u', w')$ and $\boldsymbol{g}' \equiv (0, -g)$. We non-dimensionalize these equations by introducing the following scales: h_0, the undisturbed depth in $x' < 0$; a, a typical wave amplitude; λ, a typical wavelength; see figure 1. The appropriate speed scale is $(gh_0)^{1/2}$, which is the basis for defining a timescale; we now obtain the familiar non-dimensional equations

$$u_t + \epsilon(uu_x + wu_z) + p_x = 0; \quad \delta^2\{w_t + \epsilon(uw_x + ww_z)\} + p_z = 0; \quad u_x + w_z = 0, \tag{1}$$

with $1 - z + \epsilon p = 0$, $w = \eta_t + \epsilon u\eta_x$ on $z = 1 + \epsilon\eta$; $w/u = \mathrm{d}b/\mathrm{d}x$ on $z = b(x)$. Here we have written $p(x, z, t)$ as the deviation from the undisturbed hydrostatic pressure distribution, and the parameters are defined by $\epsilon = a/h_0$, $\delta = h_0/\lambda$. The free surface is represented by $z = 1 + \epsilon\eta(x, t)$.

In addition, the equation of mass conservation, together with the boundary conditions on w, imply the familiar result

$$\frac{\partial\eta}{\partial t} + \frac{\partial\bar{u}}{\partial x} = 0, \quad \bar{u} = \int_b^{1+\epsilon\eta} u(x, z, t)\,\mathrm{d}z.$$

Further, if undisturbed conditions exist far enough ahead of and behind the wave, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \eta(x, t)\,\mathrm{d}x = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \eta(x, t)\,\mathrm{d}x = \text{constant}, \tag{2}$$

which is the otherwise obvious conservation of mass associated with the surface wave. We shall suppose that the wave is generated at the left (where the depth is constant) and that it is prescribed; it therefore carries a known mass, the constant in (2).

Our equations contain the conventional parameters encountered in water-wave theory; we now proceed with the choice of variables

$$\chi = \epsilon^{1/2}x/\delta, \quad \tau = \epsilon^{1/2}t/\delta$$

(and $w \to \epsilon^{1/2}w/\delta$). This transformation is used in conjunction with the variation of depth governed by the function $b(x) = B(\alpha x)$ where α^{-1} is the scale on which the depth changes. There are then three problems that are of interest, each controlled by the size of α in relation to ϵ. The choices are

$$(a)\ \delta\alpha/\epsilon^{1/2} = \epsilon\sigma; \quad (b)\ \delta\alpha/\epsilon^{1/2} = \epsilon; \quad (c)\ \delta\alpha/\epsilon^{1/2} = \sigma, \quad \epsilon = \Delta\sigma,$$

for $\epsilon \to 0$, $\sigma \to 0$, $\Delta \to 0$. Case (a) describes the situation in which the depth change occurs on a scale longer than that associated with the evolution of the nonlinear wave. This is the problem for which most work has been done; it is, of course, the one which corresponds to the perturbation of the IST. Case (b) is the one in which the two scales are the same and then we obtain the 'true' variable-depth KdV equation (see Johnson 1973a). This is the one case for which we can make no analytical headway (at least, not for arbitrary depth $D(Y) = 1 - B(Y)$, $Y = \alpha x$). Finally we have case (c), which is the situation where the depth changes slowly but faster than the evolution of the nonlinear wave. It is our intention to describe some results for cases (a) and (c), but without invoking any further assumptions about the relationship between ϵ and σ, or between σ and Δ. That is, we treat $\epsilon \to 0$ and $\sigma \to 0$ (and similarly σ, Δ) as independent parameters throughout our calculations.

Finally, we introduce appropriate variables that are relevant to the various propagation modes in this problem. The primary wave ($\eta = O(1)$), which initiates the whole process, propagates from left to right; thus we require the right-going characteristic and suitable (slow) scales for the wave's evolution (and the description of other right-going components). For case (a) these are

$$\xi = \frac{1}{\epsilon\sigma}\int_0^Y R(y)\,\mathrm{d}y - \tau, \quad X = \frac{1}{\sigma}\int_0^Y S(y)\,\mathrm{d}y, \quad Y = \epsilon\sigma\chi, \tag{3}$$

where the functions $R(Y)$ and $S(Y)$ are to be determined (in terms of $D(Y)$). The left-going wave (the reflection) evolves slowly with respect to the left-going characteristic, the scale being given by the scale on which the depth changes; thus we introduce

$$\zeta = \int_0^Y R(y)\,\mathrm{d}y + T,$$

where $T = \epsilon\sigma\tau$. The solution for the surface wave will therefore be expressed in terms of the variables (ξ, ζ, X, Y) and the parameters (ϵ, σ). The governing equations now become

$$\begin{aligned}\epsilon\sigma u_\zeta - u_\xi + \epsilon u\{R(u_\xi + \epsilon\sigma u_\zeta) + \epsilon S u_X + \epsilon\sigma u_Y\}&\\ + \epsilon w u_z + R(p_\xi + \epsilon\sigma p_\zeta) + \epsilon S p_X + \epsilon\sigma p_Y &= 0; \end{aligned}\tag{4a}$$

$$\epsilon\{\epsilon\sigma w_\zeta - w_\xi + \epsilon u[R(w_\xi + \epsilon\sigma w_\zeta) + \epsilon S w_X + \epsilon\sigma w_Y] + \epsilon w w_z\} + p_z = 0; \tag{4b}$$

$$R(u_\xi + \epsilon\sigma u_\zeta) + \epsilon S u_X + \epsilon\sigma u_Y + w_z = 0; \tag{4c}$$

with

$$\left.\begin{aligned} p &= \eta \qquad (4d)\\ w &= \epsilon\sigma\eta_\zeta - \eta_\xi + \epsilon u\{R(\eta_\xi + \epsilon\sigma\eta_\zeta) + \epsilon S\eta_X + \epsilon\sigma\eta_Y\} \qquad (4e)\end{aligned}\right\} \text{ on } z = 1 + \epsilon\eta,$$

and

$$w = -\epsilon\sigma D'(Y)u \quad \text{on} \quad z = 1 - D(Y). \tag{4f}$$

The corresponding choices for case (c) are

$$\xi = \frac{1}{\sigma}\int_0^Y R(y)\,\mathrm{d}y - \tau, \quad X = \Delta\int_0^Y S(y)\,\mathrm{d}y, \quad Y = \sigma\chi; \zeta = \int_0^Y R(y)\,\mathrm{d}y + T,$$

where $T = \sigma\tau$ and $\Delta\sigma = \epsilon$. (It is convenient to use the same symbols here as for case

(a).) The governing equations in this case follow directly: they are equations (4) with $\epsilon\sigma$ replaced by σ, and ϵ replaced by $\Delta\sigma$. We seek a solution of these equations by writing all the dependent variables as (double) asymptotic expansions in the form

$$Q \sim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon^n \sigma^m Q_{nm}, \quad \epsilon \to 0, \quad \sigma \to 0$$

(for case (a)). Here Q represents each of u, w, p and η; for case (c) the same structure is employed but with ϵ replaced by Δ.

3. Case (a): $Y = \epsilon\sigma\chi$

This is the case for which the general structure of the solution is now well-understood (mainly from the work of Knickerbocker & Newell 1985) although we shall describe some new details; we present the salient features of this problem in figure 2. This figure depicts the wave fronts and wave components associated with the transport of $O(1)$ mass. To proceed, we construct the system of equations at each order, $\epsilon^n\sigma^m$, and impose the conditions that ensure uniform asymptotic expansions. We list the main results that are obtained at each order (and we use information from previous orders, as necessary).

$\epsilon^0\sigma^0$: $R(Y) = 1/(D(Y))^{1/2}$, so that the right characteristic is $\xi = \dfrac{1}{\epsilon\sigma}\displaystyle\int_0^Y \frac{\mathrm{d}y}{(D(y))^{1/2}} - \tau$;

$$\epsilon^1\sigma^0: \qquad 2SD^{1/2}\eta_{00X} + \frac{3}{D}\eta_{00}\,\eta_{00\xi} + \frac{D}{3}\eta_{00\xi\xi\xi} = 0; \tag{5}$$

$$\epsilon^1\sigma^1: \qquad 2SD^{1/2}\eta_{01X} + \frac{3}{D}(\eta_{00}\,\eta_{01})_\xi + \frac{D}{3}\eta_{01\xi\xi\xi} = -2D^{1/4}(D^{1/4}\eta_{00})_Y, \tag{6}$$

and then with equation (5) this yields

$$D^{1/2}\int_{-\infty}^{\infty} \eta_{00}^2\,\mathrm{d}\xi = \text{constant}, \tag{7}$$

which may be interpreted as the conservation of energy for the primary wave (and, as such, could have been imposed *a priori*). We find that to remove the non-uniformities we require $S(Y) = D^{-5/2}(Y)$, and so the primary wave takes the form

$$\eta_{00} = \frac{1}{D}F_{00}(\hat{\xi}, \hat{X}); \quad (\hat{\xi}, \hat{X}) = D^{-3/2}(\xi, X) \quad \text{with} \quad 2F_{00\hat{X}} + 3F_{00}\,F_{00\hat{\xi}} + \tfrac{1}{3}F_{00\hat{\xi}\hat{\xi}\hat{\xi}} = 0; \tag{8}$$

$$\epsilon^1\sigma^2: \qquad \frac{2}{D^2}\eta_{02X} + \frac{3}{D}(\eta_{00}\,\eta_{02})_\xi + \frac{D}{3}\eta_{02\xi\xi\xi} = -\frac{3}{D}\eta_{01}\,\eta_{01\xi} - 2D^{1/4}(D^{1/4}\eta_{01})_Y; \tag{9}$$

$$\epsilon^2\sigma^2: \qquad U_{11\zeta} + \frac{1}{D^{1/2}}H_{11\zeta} + H_{11Y} = 0; \quad U_{11\zeta} + \frac{1}{D^{1/2}}H_{11\zeta} + \frac{1}{D^{1/2}}(DU_{11})_Y = 0, \tag{10}$$

or
$$H_{11T} + (DU_{11})_Y = 0; \quad U_{11T} + H_{11Y} = 0, \tag{11}$$

where the components at $O(\epsilon\sigma)$ are defined by writing

$$\eta_{11} = h_{11}(\xi, X, Y) + H_{11}(\zeta, X, Y), \quad u_{11} = u_{11}(\xi, X, Y) + U_{11}(\zeta, X, Y),$$

and with
$$2D^{1/4}(D^{1/4}h_{11\xi})_Y = -(D\eta_{00Y})_Y \tag{12}$$

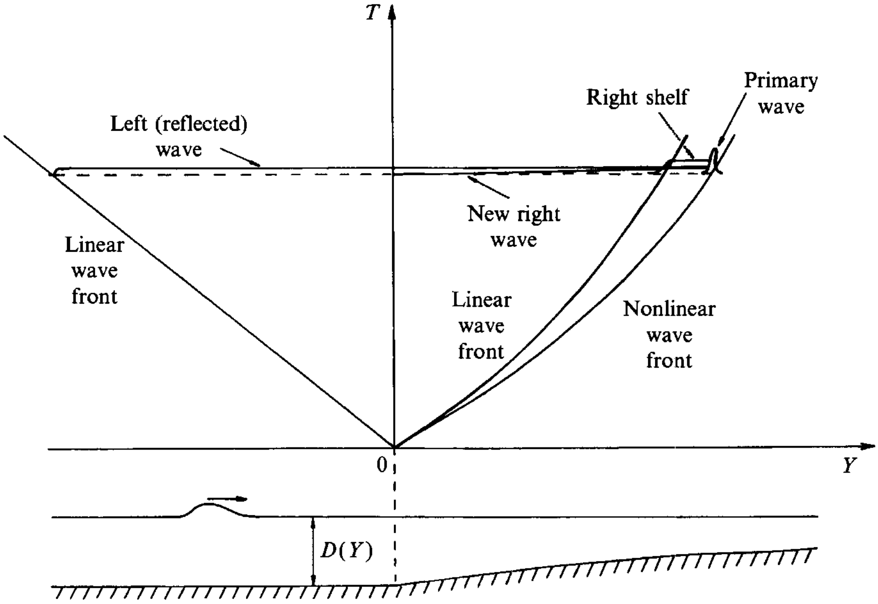

FIGURE 2. Sketch of the general structure of the solution: the components which carry $O(1)$ mass.

for the non-local contribution to η_{11}, i.e. away from the primary wave; this is a perturbation of the primary wave generated by the depth change, a component which appears to have been overlooked by previous authors. In passing, we observe that (10), upon the elimination of U_{11}, yields the single equation

$$2D^{1/4}(D^{1/4}H_{11\zeta})_Y + (DH_{11Y})_Y = 0, \tag{13}$$

which is to be compared with (12). (The apparent novel aspects of this solution are easily recovered from the model equations

$$D_- u + \frac{1}{D^{1/2}} D_+ p + \epsilon\sigma p_Y = 0; \quad p_z = 0; \quad \frac{1}{D^{1/2}} D_+ u + \epsilon\sigma u_Y + w_z = 0,$$

with $p = \eta$ and $w = D_- \eta$ on $z = 1$; $w = -\epsilon\sigma u D'$ on $z = 1 - D$, where

$$D_\pm \equiv \pm\partial/\partial\xi + \epsilon\sigma\,\partial/\partial\zeta.$$

This problem ignores the contribution from the nonlinearity, evolution (in X) and dispersion.) We see that h_{11} is forced by both the primary wave, η_{00}, and a variation in the depth. This term therefore represents a contribution to the distortion of the primary wave as it enters a region of variable depth. The other component, H_{11}, satisfies a homogeneous equation and consequently is not forced in the same way; we shall see that it exists to maintain the overall mass balance, to leading order.

We now suppose (for simplicity) that η_{00} is a solitary wave. Let the peak of this wave be at $\theta = \xi - cX = 0$, where c is the (constant) speed of the wave in (ξ, X)-space and, further, let $\theta = 0$ be at $Y = \bar{Y}(T)$. The equation for η_{01}, (6), then becomes

$$\frac{2}{D^2}(\eta_{01X} - c\eta_{01\theta}) + \frac{3}{D}(\eta_{00}\,\eta_{01})_\theta + \frac{D}{3}\eta_{01\theta\theta\theta} = -2D^{1/4}(D^{1/4}\eta_{00})_Y,$$

and when we construct the integral of η_{01} over θ, in an appropriate neighbourhood of $\theta = 0$, we confirm the existence of a shelf. Further, from (9), we find that $D^{1/4}\eta_{01} =$

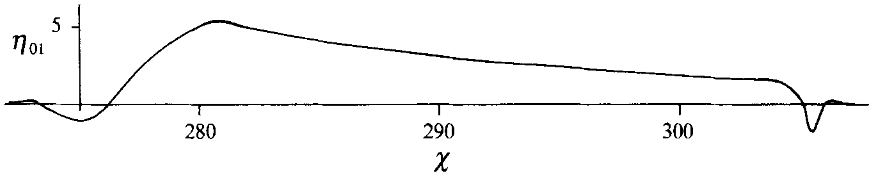

FIGURE 3. The right-going shelf behind the solitary wave of initial amplitude $A = 1$. The depth variation is $(1-Y)^{4/3}$, terminating at $D = \frac{1}{2}$ ($\chi \approx 405$); $\epsilon = 0.1$, $\sigma = 0.01$ and the solitary wave is three-quarters of the way up the slope ($\chi \approx 304$).

constant on the shelf; it is therefore convenient to write $\eta_{01} = D^{-1/4}S_{01}$, which is usually regarded as Green's law, and then the shelf amplitude is found to be

$$S_{01} = -\frac{3m_0}{4c}\bar{D}^{9/4}\bar{D}', \; \bar{D} = D(\bar{Y}).$$

In the case of the solitary wave of initial amplitude A, for which $m_0 = 4(A/3)^{1/2}$ and $c = A/2$, we recover the result given by Knickerbocker & Newell (1985); we see that $S_{01} > 0$ if $\bar{D}' < 0$.

The shelf exists where η_{00} is exponentially small and so, with $\eta_{01} = D^{-1/4}F_{01}$, we can then show that

$$F_{01} \sim \mathscr{F}'\left(\sigma\theta + c\int_0^T \bar{D}^{-2}\,\mathrm{d}T\right),$$

where

$$\mathscr{F}(\mathscr{T}) = -m_0\bar{D}^{3/4} + \text{constant}, \quad \bar{D} = \bar{D}(\mathscr{T}), \mathscr{T} = c\int_0^T \bar{D}^{-2}\,\mathrm{d}T.$$

At the front of the right-going shelf, a transition extends over a distance ($\theta = O(1)$) to either side of $\theta = 0$, between the front and the rear of the solitary wave. Similarly, a rear transition occurs near $\xi = 0$, over a distance $\xi = O(1)$; the appropriate asymptotic solutions are readily obtained from our equation for η_{01}.

All these phenomena have been obtained, at various stages, by previous authors working from the perturbated-IST problem (e.g. Kaup & Newell 1978; Knickerbocker & Newell 1980) which is, of course, equivalent to our equation (6). We comment that this (right) shelf is part of the nonlinear structure, in the sense that its detailed form depends crucially on the nature of the primary wave via the perturbation of the IST (although our approach is more direct); see Kaup & Newell (1978).

Some numerical results, based on our analysis of the right-going shelf (together with its transitions), are presented in figure 3. These calculations make use of a depth change which represents a $\frac{4}{3}$ power-law transition from $D = 1$ to $D = \frac{1}{2}$. The right-going shelf is depicted when the solitary wave is three-quarters of the way 'up' the slope. (These results are readily extended to two-soliton solutions, for example, where two shelves now appear.)

4. Mass conservation

So far, it has been unnecessary to invoke the explicit statement of mass conservation, (3). However, in order to proceed, we must now examine

$$\int_{-\infty}^{\infty} \eta\,\mathrm{d}\chi = \int_{-\infty}^{\infty} (\eta_{00} + \sigma\eta_{01} + \epsilon\eta_{10} + \epsilon\sigma\eta_{11} + \ldots)\,\mathrm{d}\chi = \text{constant}\,(= O(1)),$$

and determine all those contributions which carry $O(1)$ mass. First we have

$$\int_{-\infty}^{\infty} \eta_{00}\,\mathrm{d}\chi \sim \bar{D}\int_{-\infty}^{\infty} F_{00}(\hat{\xi},\hat{X})\,\mathrm{d}\hat{\xi} = m_0\bar{D},$$

where $\bar{D} = D(\bar{Y})$, with $Y = \bar{Y}(T)$ on $\hat{\xi} = 0$ (for $\epsilon \to 0$), and m_0 is the mass associated with the primary wave which propagates from the left. Of course, in $\bar{Y} < 0$ we have $\bar{D} = 1$, but as the depth varies we see that the resulting distortion of the wave leads to a change in the mass that it carries; this effect must be accommodated by other wave components.

The mass carried by the right-going shelf is easily seen to be

$$\sigma\int_{-\infty}^{\infty} \eta_{01}(\xi, X, Y)\,\mathrm{d}\chi \sim m_0(\bar{D}^{1/4} - \bar{D})$$

which incorporates the property that this mass must be zero in $\bar{Y} < 0$ (where $\bar{D} = 1$).

The total mass so far is therefore

$$\int_{-\infty}^{\infty} (\eta_{00} + \sigma\eta_{01})\,\mathrm{d}\chi = m_0\bar{D}^{1/4} + O(\epsilon), \tag{14}$$

for which the leading term is $O(1)$ but not constant. These results confirm the observations made by Knickerbocker & Newell (1980, 1985) and by Miles (1979), which have led to the important conclusion that a left-going (or at least appropriate components of η_{11}) must carry the mass

$$m_0 - m_0\bar{D}^{1/4} \tag{15}$$

to leading order, so that the total mass satisfies

$$\int_{\infty}^{\infty} \eta\,\mathrm{d}\chi \sim m_0 \quad \text{for all} \quad X, Y.$$

As to the other contributions to η, it is not too difficult to confirm that $\epsilon\eta_{10}$ generates a mass of $O(\epsilon)$, and that all higher-order terms produce correspondingly smaller contributions (as $\epsilon \to 0$, $\sigma \to 0$).

We now examine the mass carried by the right-going component of η_{11}. This wave, $h_{11}(\xi, Y, X)$, satisfies equation (12) and can therefore be integrated directly to yield

$$h_{11} = \tfrac{1}{2}D^{-1/4}\int_0^Y D^{-1/4}\frac{\partial}{\partial Y}\left\{D\int_{\xi}^{\infty} \eta_{00Y}\,\mathrm{d}\xi\right)\mathrm{d}Y. \tag{16}$$

The solution has been chosen to satisfy the condition of no disturbance ahead of the primary wave, so $h_{11} \to 0$ as $\xi \to +\infty$. Furthermore, no such solution can exist in $Y < 0$ and, indeed, is initiated by the depth change in $Y \geqslant 0$; thus this solution also satisfies $h_{11} \to 0$ as $Y \to 0^+$. This wave, which is essentially a linear phenomenon, is a perturbation to the primary wave, generated as the solitary wave enters a region of variable depth; further, because it also arises at the same order as the left (reflected) wave, h_{11} can be regarded as a re-reflection. In the context of our problem, the important question is whether this component carries $O(1)$ mass; it follows directly that

$$\epsilon\sigma\int_{-\infty}^{\infty} h_{11}\,d\chi \sim \frac{m_0}{4}\int_0^{\bar{Y}(T)} D^{-1/4}\left\{\int_0^Y D^{-1/4}(D^{1/2}D')'\,\mathrm{d}y\right\}\mathrm{d}Y, \tag{17}$$

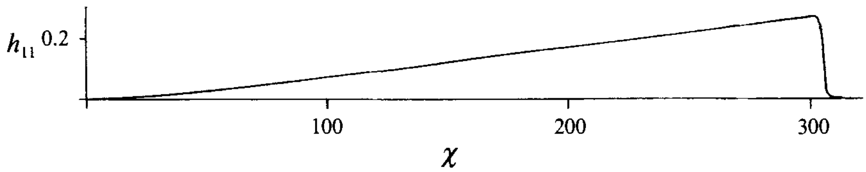

FIGURE 4. The new right-going shelf generated by the solitary wave of initial amplitude $A = 1$. The depth variation is $(1-Y)^{4/3}$, terminating at $D = \frac{1}{2}$ ($\chi \approx 405$); $\epsilon = 0.1$, $\sigma = 0.01$ and the solitary wave is three-quarters of the way up the slope ($\chi \approx 304$).

where the primary wave is at $\theta = 0$ (i.e. $Y = \bar{Y}(T)$), as $\epsilon \to 0$. Thus the right-going component of η_{11} does indeed carry $O(1)$ mass (except in the special case $D = (1+\alpha Y)^{2/3}$ for which $(D^{1/2}D')' = 0$; then the $O(1)$ contribution in (17) is identically zero).

If the primary wave, η_{00}, is a solitary wave with its peak at $Y = \bar{Y}(T)$, and we use the depth variation $D(Y) = (1+\alpha Y)^{4/3}$, then the predominant behaviour of this right-going component is

$$h_{11} = \tfrac{1}{2}\alpha m_0(D^{1/4} - D^{-1/4}), \quad 0 \leqslant Y < \bar{Y} < Y_0;$$

$Y = Y_0$ is where D again becomes constant. The complete solution for h_{11}, from (16), also incorporates a transition structure near $Y = \bar{Y}$. The total mass carried by this component is

$$\tfrac{3}{8}m_0(1+\bar{D}-2\bar{D}^{1/2}), \quad 0 < \bar{Y} < Y_0, \tag{18}$$

where $\bar{D} = D(\bar{Y})$. An example of this wave component is given in figure 4.

The left-going component of η_{11}, that is H_{11}, must provide (for $\bar{Y} < Y_0$, say) the mass

$$M_0(T) = m_0 - m_0\,\bar{D}^{1/4} - \tfrac{3}{8}m_0(1+\bar{D}-2\bar{D}^{1/2}), \tag{19}$$

from (15) and (18). Now H_{11} cannot exist forward of the primary wave other than in the form of a transition. In this discussion we shall restrict the analysis to the case of a solitary wave centred at $\theta = 0$; the shelf is therefore limited to the right by $\theta = 0$. To the left it cannot propagate further than the (linear) leftward characteristic emanating from $Y = 0$ (see figure 2). The left-going shelf is therefore restricted to lie between $\zeta = 0$ ($Y = \hat{Y}(T)$ say) and $\theta = 0$ ($Y = \bar{Y}(T)$). (The transitions that return the shelf to undisturbed conditions near $Y = \hat{Y}$ and $Y = \bar{Y}$ are readily obtained from our equations.) The appropriate solution for H_{11} (and U_{11}) is obtained only when the mass condition, (19), is imposed; thus we shall require an expression for

$$\epsilon\sigma\int_{-\infty}^{\infty} H_{11}(\zeta, Y)\,\mathrm{d}\chi = \int_{-\infty}^{\infty} H_{11}\left(T + \int_0^Y \frac{\mathrm{d}y}{D^{1/2}}, Y\right)\mathrm{d}Y.$$

This can be derived directly from (11) to yield

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{\hat{Y}}^{\bar{Y}} H_{11}\,\mathrm{d}Y + \hat{H}_{11}\frac{\mathrm{d}\hat{Y}}{\mathrm{d}T} - \bar{H}_{11}\frac{\mathrm{d}\bar{Y}}{\mathrm{d}T} + \bar{D}\bar{U}_{11} - \hat{D}\hat{U}_{11} = 0;$$

here the circumflex denotes evaluation on $Y = \hat{Y}$, and the overbar evaluation on $Y = \bar{Y}$. But we have

$$\int_{\hat{Y}}^{\bar{Y}} H_{11}\,\mathrm{d}Y \sim M_0(T); \quad \frac{\mathrm{d}\hat{Y}}{\mathrm{d}T} = -1; \quad \frac{\mathrm{d}\bar{Y}}{\mathrm{d}T} \sim \bar{D}^{1/2} \quad \text{and} \quad \hat{D} = 1,$$

and so we obtain

$$\bar{D}\bar{U}_{11}-\bar{D}^{1/2}\bar{H}_{11}-(\hat{H}_{11}+\hat{U}_{11}) = \tfrac{1}{4}m_0\frac{\bar{D}'}{\bar{D}^{1/4}}+\tfrac{3}{8}m_0(\bar{D}^{1/2}-1)\bar{D}', \tag{20}$$

to leading order; the first term on the right (associated with $\frac{1}{4}m_0$) alone corresponds to the result obtained by Knickerbocker & Newell (1985). We now require the solution of (11), subject to the condition (20), given the depth variation in $Y>0$.

One of our aims in this work is to extract as much analytical detail as we reasonably can. To this end, we proceed with the solution of this problem with the choice (mentioned earlier)

$$D(Y) = (1+\alpha Y)^{4/3}, \quad Y>0, \tag{21}$$

where α is a constant. There is nothing remarkable about this depth variation, either for $\alpha>0$ or $\alpha<0$: indeed, it could be used to model changes that are not far from linear (and a linear depth variation is often used to exemplify this problem). For us, the importance of (21) is that it leads immediately to the general solution of (11) as

$$H_{11} = \begin{cases} -(6/\alpha)\,G''(\zeta), & Y\leqslant 0, \\ D^{-1/4}\{-(6/\alpha)\,G''(\zeta)-G'(\zeta)+G'(\tilde{\xi})\}, & Y>0, \end{cases} \tag{22}$$

where

$$G'(\zeta) = -\tfrac{1}{6}\alpha m_0\,\mathrm{e}^{-\alpha\zeta/6}+\tfrac{1}{8}\alpha m_0(1-\tfrac{1}{6}\alpha\zeta^2)-\tfrac{1}{4}\alpha m_0\ln(1+\tfrac{1}{6}\alpha\zeta)$$
$$+\tfrac{1}{4}\alpha m_0\,\mathrm{e}^{-(1+\alpha\zeta/6)}\int_1^{1+\alpha/6} Y^{-1}\,\mathrm{e}^{-Y}\,\mathrm{d}Y+\text{constant}.$$

This expression, and the corresponding one for $G''(\zeta)$, are used directly in (22) to produce the final result for H_{11}. (The arbitrary constant which appears in $G'(\zeta)$ cancels identically in H_{11}.) A similar expression for U_{11} can also be obtained.

Our results are to be compared with those of Knickerbocker & Newell (1985), where the structure of this left shelf was touched on only through numerial integrations (for a linear depth change). As mentioned by these authors, this solution is not the conventional Green's law; this is usually regarded as the property: $H_{11}\propto D^{-1/4}$ and $U_{11}\propto D^{-3/4}$. This does not occur here because the evolution of this left-going wave is on precisely the same scale as that on which D changes. (In fact, this would seem to indicate an area of confusion in the analysis presented by Knickerbocker & Newell 1985: they explain that these two scales are the same, but then use (in our notation)

$$H_{11_T}-D^{1/2}H_{11_Y} = 0.$$

This is clearly not true unless $D'(Y)$ can be neglected, which is possible with these scales only if $D=$ constant. The correct equation for H_{11} is our equation (13); the solution does not yield $H_{11}=$ constant nor $DU_{11}=$ constant, on left-going characteristics. A direct comparison with their work is possible if we ignore the right-going component, h_{11}; the left-going shelf, for our depth change, is then simply

$$H_{11} = \begin{cases} -\tfrac{1}{6}m_0\,\alpha\,\mathrm{e}^{-\alpha\zeta/6}, & \hat{Y}\leqslant Y\leqslant 0, \\ -\tfrac{1}{6}m_0\,\alpha D^{-1/4}\,\mathrm{e}^{-\alpha\zeta/6}\exp(D^{1/4}-1), & 0<Y\leqslant\bar{Y}, \end{cases} \tag{23}$$

with $U_{11}=(D^{-1/2}-2D^{-3/4})H_{11}$, where $D(Y)$ is given by (21) in $Y>0$ and $D=1$ for $Y\leqslant 0$.)

Finally, we mention the form that the solution takes if the primary wave enters another region of constant depth ($D=D_0$). In this case, the condition satisfied by H_{11}

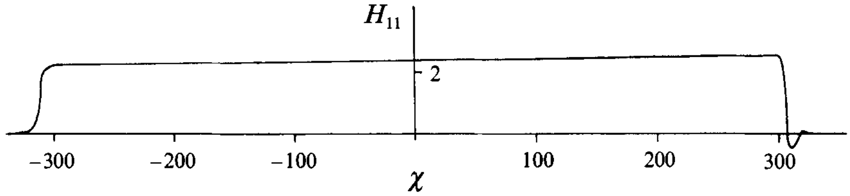

FIGURE 5. The left-going (reflected) wave (described by (23)) generated by a solitary wave of initial amplitude $A = 1$. The depth variation is $(1-Y)^{4/3}$, terminating at $D = \frac{1}{2}$ ($\chi \approx 405$); $\epsilon = 0.1$, $\sigma = 0.01$ and the solitary wave is three-quarters of the way up the slope ($\chi \approx 304$).

and U_{11} is (20), but with an alternative expression replacing that used in (19); this yields, in $Y > Y_0$, $\eta_{11} = 0$. The whole wave structure now evolves in the obvious manner: the primary wave moves away from the right-going components that are propagating at the linear wave speed. These components (the right shelf and the perturbation, h_{11}) move together to the right, producing a shelf-like structure moving rightwards. Correspondingly, the left-going shelf (H_{11}) propagates to the left (so that its right end moves to the left into $Y < Y_0$). These various components together carry the totality of the leading-order mass brought by the primary wave from the left. A particular left shelf, described by (23) and with appropriate transitions, is shown in figure 5.

5. Case (c): $Y = \sigma\chi$, $\epsilon = \Delta\sigma$

The variable depth now varies on a scale which is shorter than the scale associated with the wave evolution. Here an important difference from case (*a*) should be recognized: the right shelf arises at $O(\sigma)$ in that case, i.e. {depth variation scale $(\epsilon\sigma)$}/{wave evolution scale (ϵ)}. The corresponding result in this case yields $O(\sigma)/O(\Delta\sigma) = O(\Delta^{-1})$ which cannot occur in the asymptotic solution here. Hence we anticipate that a right-going shelf – or at least one that carries $O(1)$ mass – will not appear in this problem in the same way that η_{01} did before.

We follow the same procedure as adopted for case (*a*), but here we limit ourselves to a statement of the results obtained. First, the characteristic variables are again defined by the familiar requirement that $R(Y) = (D(Y))^{-1/2}$, and then the primary wave takes the form

$$\eta_{00} = D^{1/4} f_{00}(\xi, X) \quad \text{where} \quad 2f_{00X} + 3f_{00}f_{00\xi} + \tfrac{1}{3}f_{00\xi\xi\xi} = 0.$$

This KdV equation has been written with the choice $S = 1$ for all Y, since the conservation of energy is automatically satisfied here (cf. (7)).

The mass carried by the primary wave is

$$\int_{-\infty}^{\infty} \eta_{00}\,\mathrm{d}\chi = \int_{-\infty}^{\infty} D^{-1/4} f_{00}\,\mathrm{d}\chi \sim \bar{D}^{1/4} \int_{-\infty}^{\infty} f_{00}\,\mathrm{d}\xi = m_0 \bar{D}^{1/4}, \tag{24}$$

since $\mathrm{d}\chi/\mathrm{d}\xi = D^{1/2}$ and the wave is at $\xi = 0$ (to leading order) where $Y = \bar{Y}$. The mass m_0 is that associated with η_{00} in $Y < 0$ where $\bar{D} = 1$, just as for case (*a*). Now the mass given in (24) is precisely that carried by the primary wave plus the right shelf in our previous calculation; see (14). Here, therefore, there is no right-going shelf (of amplitude greater than $O(\sigma)$) which carries $O(1)$ mass, in agreement with our earlier scaling argument.

The contribution to the right-going mass from η_{10} is easily demonstrated to be $o(\Delta)$.

However, the presence of additional wave components is indicated in order to compensate for the mass imbalance, namely $m_0 - m_0 \bar{D}^{1/4}$, exactly as in case (*a*). We write

$$\eta_{01} = h_{01}(\xi, X, Y) + H_{01}(\zeta, X, Y),$$

and then we find that

$$2D^{1/4}(D^{1/4}H_{01\zeta})_Y + (DH_{01Y})_Y = 0, \tag{25}$$

and (see (12))

$$2D^{1/4}(D^{1/4}h_{01\xi})_Y = -(D\eta_{00Y})_Y$$

so that

$$h_{01} = -\tfrac{1}{8}D^{-1/4}\left\{\int_0^Y D^{-1/4}(D^{-1/4}D')'\,\mathrm{d}Y\right\}\left\{\int_\xi^\infty f_{00}\,\mathrm{d}\xi\right\}, \tag{26}$$

which is to be compared with (16). It follows that both h_{01} and H_{01} carry $O(1)$ mass, where H_{01} is constructed to ensure that $O(1)$ mass is conserved.

In case (*a*) we presented some detailed results for a special depth variation and we may proceed along the same lines here. However, this particular depth variation satisfies $(D'D^{-1/4})' = 0$ and so the right-going component, (26), is identically zero. This special case corresponds to $(D'D^{1/2})' = 0$ which, in case (*a*), led to the absence of that right-going wave (h_{11}). In consequence, the left-going wave – the shelf – is required to satisfy (25) and to accommodate the mass (15). But this is precisely the problem that led to the solution (23) in case (*a*), and therefore that solution is the relevant one here; the form of this shelf is shown in figure 5. The description of the wave components that are present when the primary wave enters a region of constant depth are then exactly as for case (*a*).

6. Discussion

The work presented here has attempted to give a reasonably complete description of the effects of slowly varying depth on weakly nonlinear surface waves. Our approach has been to work directly with the appropriate governing equations that arise at each order. We have described the right-going shelf and its evolution, and we have shown how the left-going wave – the reflection – and a right-going perturbation of the primary wave are straightforwardly derived, particularly for the special depth variation $D = (1+\alpha Y)^{4/3}$. The detail evident in our results is to be compared with that obtained by other authors (particularly Miles 1979, and Knickerbocker & Newell 1980, 1985).

In case (*c*) the depth variation is slow, but on a scale faster than that associated with the evolution of the (nonlinear) primary wave. An important observation that we have made for this problem is that the mass carried by the primary wave ($m_0 \bar{D}^{1/4}$) is the same as that carried by the primary wave plus right-going shelf in case (*a*). This confirms the conjecture, based on scaling arguments, that no right shelf (like that produced behind the primary wave in (*a*)) is possible for problem (*c*). In other aspects, though, the general structure of the solution is essentially the same in both cases: an evolving primary wave, a reflected wave and the new right-going component.

We have used a special depth variation in order to explain more fully the character of the various wave components. The corresponding details for case (*c*) have also been developed; in both cases we have described the appearance of a new right-going perturbation of the primary wave, which is initiated by the depth change. This wave, which is generated from $Y = 0$, has the general appearance of a shelf and, most importantly, it can carry $O(1)$ mass. (Another special choice, $D(Y) = (1+\alpha Y)^{8/5}$, also leads to a fairly simple solution but the final results for the wave components differ only in the finest detail and are therefore not discussed here.) Nevertheless two depth profiles could bear closer scrutiny, which might then lead to a deeper understanding of the evolution of the various components. These two are: profiles close to $(1+\alpha Y)^{2/3}$ for case (a), and close to $(1+\alpha Y)^{4/3}$ for case (c); these are the ones for which the right-going perturbation is absent. The mechanism underlying this wave's appearance/disappearance (at this order) for various $D(Y)$ is an investigation of some interest; this is put aside for later study.

We have mentioned the role of perturbed-IST theory to the problem of case (a), but some other ideas from soliton theory are also relevant to case (c). An important phenomenon associated with a solitary wave (for example) which moves into a region of reduced depth (provided the depth change is rapid enough) is the fission of solitons; see Tappert & Zabusky (1971), Johnson (1973*b*). It is easily demonstrated that this is predicted from our equations.

Many simplifying assumptions have been incorporated here in order to emphasize the main features of the wave phenomena that we have described. Clearly it is possible to extend this work by relaxing at least some of the assumptions and examining the consequences. The obvious improvements, such as the inclusion of an underlying shear flow and the extension to two-dimensional surface waves, are now under investigation.

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